Transitive graphs in counterexamples to Karp's conjecture

Alexander Engström*

Institute of Theoretical Computer Science,
ETH Zürich, CH-8092 Zürich, Switzerland

February 2, 2008

Abstract: Karp conjectured that all nontrivial monotone graph properties are evasive. This was proved for n a prime power, and $n = 6$, where n is the number of graph vertices, by Kahn, Saks, and Sturtevant. We give a complete description of which transitive graphs are contained in a possible counterexample when $n = 10$.

1 Introduction

The notion of evasiveness came from the study of argumented complexity, but with Kahn, Saks and Sturtevants influential paper [2] it was incorporated in combinatorial and algebraic topology. A graph property is a partition of the unlabeled graphs into two classes, those with and those without the property. If the property is preserved under the removal of edges it is monotone. Let us fix a monotone graph property and the number of graph vertices, then we can create a simplicial complex with the graph edges as vertices and the graphs with the property as simplices.

Example 1 *Let the graph property be planarity and use 5 graph vertices. All but the complete graph are planar, so the faces of the simplicial complex are all proper subsets of the graph edge set.*

Definition 2 *A simplicial complex Δ is nonevasive if it is a point, or if there is a vertex v of Δ such that both the link $\mathrm{lk}_\Delta(v) = \{\sigma \in \Delta \mid v \not\in \Delta, \sigma \cup \{v\} \in \Delta\}$ and the deletion $\mathrm{dl}_\Delta(v) = \{\sigma \in \Delta \mid v \not\in \Delta\}$ are nonevasive.*

A simplicial complex is evasive if it is not nonevasive and a monotone graph property on a certain number of vertices is evasive or nonevasive dependent on

*Research supported by ETH and Swiss National Science Foundation Grant PP002-102738/1. E-mail: engstroa@inf.ethz.ch

its simplicial complex. A trivial graph properties include all or none graphs. Now we can state the famous conjecture by Karp.

Conjecture 3 *All nontrivial monotone graph properties are evasive.*

Kahn et al [2] used fixed-point theorems by Oliver [8] and Smith [10] and the implications (see [1, 2])

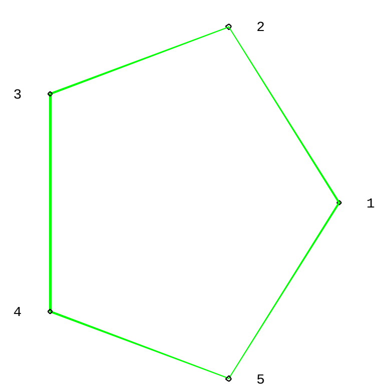
$$\text{nonevasive} \Rightarrow \text{collapsible} \Rightarrow \text{contractible} \Rightarrow \mathbb{Z} - \text{acyclic} \Rightarrow \mathbb{Z}_p - \text{acyclic}$$

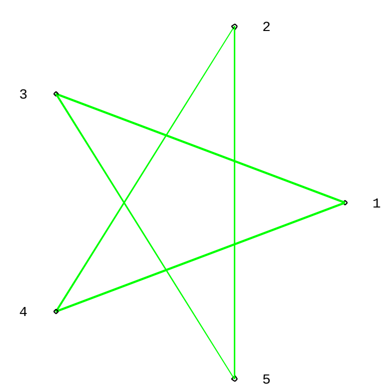
to prove the conjecture when the number of graph vertices is a prime power or 6. The goal of this paper is a characterization of which vertex transitive graphs are in a possible counterexample of the conjecture for graph properties on 10 vertices. Our method is the topological and we will use these results by Oliver [8]:

Theorem 4 *If $\Gamma' \lhd \Gamma$, Γ/Γ' is cyclic, Γ' is of p prime power order, and Δ is $\mathbb{Z}_p$–acyclic, then $\chi(\Delta^\Gamma) = 1$.*

Theorem 5 *If $\Gamma'' \lhd \Gamma' \lhd \Gamma$, Γ'/Γ'' is cyclic, Γ'' is of p prime power order, Γ/Γ' is of q prime power order, and Δ is $\mathbb{Z}_p$–acyclic, then $\chi(\Delta^\Gamma) \equiv 1 \bmod q$.*

Example 6 *To illustrate the method, we prove the conjecture when there are five vertices. Label the vertices $1, 2, 3, 4, 5$. Assume that the conjecture is false and create a simplicial complex Δ from that graph property. The graph property is nontrivial, so Δ is neither empty nor a full simplex. Since Δ is nonevasive it is $\mathbb{Z}$-acyclic and we can use Theorem 4. The action of the cyclic group $\Gamma = \langle(1\,2\,3\,4\,5)\rangle$ on the graph vertices, induces an action on the graph edges, which is the same as the vertices of Δ. The abstract simplicial complex Δ^Γ has the minimal nonempty Γ-invariant faces of Δ as vertex set, and a set of vertices of Δ^Γ is a face of Δ^Γ if their union is a face of Δ. The minimal nonempty Γ-invariant graphs are:*





If the five cycle is in Δ then Δ^Γ is two disjoint points, and $\chi(\Delta^\Gamma) = 2$. If not, then $\chi(\Delta^\Gamma) = 0$. Using Theorem 4 with Γ' as the trivial group we get that $\chi(\Delta^\Gamma) = 1$, which is a contradiction. Hence the conjecture is true for five vertices.

The plan of the rest of the paper is: First we describe all transitive graphs on ten vertices and their inclusion order. Then we use Theorem 4 and 5 to get conditions on which graphs can be in a counterexample of the conjecture. The inclusion order and conditions are investigated and they give six different cases for which transitive graphs are in a counterexample. As an appendix we describe some computational methods.

The reader who wants a more elaborate description of the method herein could look at the original work by Kahn et al [2] and for example Lutz [5, 6]. Nonevasiveness has poped up outside the domain of graph properties, see Kozlov [3], Kurzweil [4], and Welker [11].

2 The transitive graphs

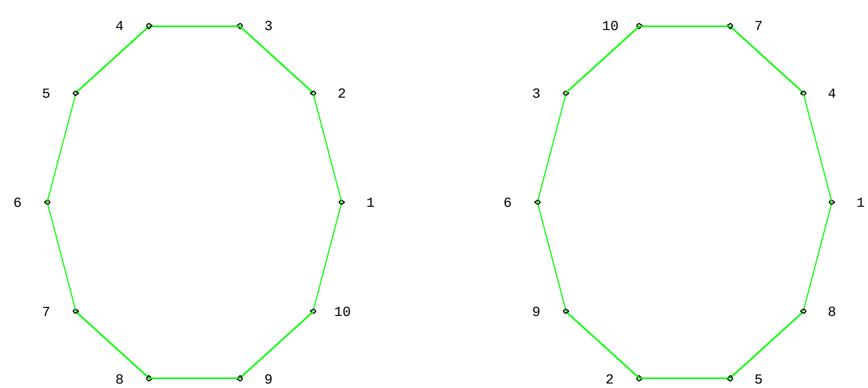
2.1 Cayley graphs

There are 22 transitive graphs on 10 vertices, and 20 of them are Cayley graphs [7, 9].

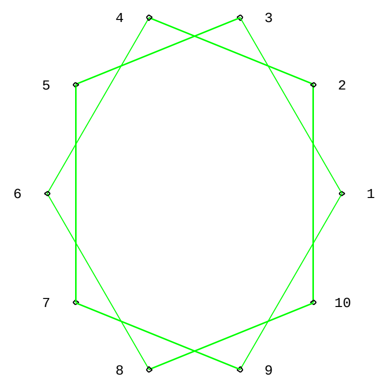
2.1.1 Definitions

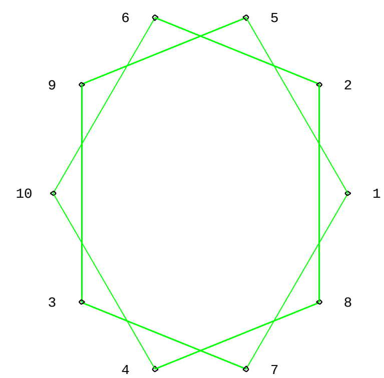
Definition 7 *Let D be a subset of $\{1,2,3,4,5\}$. The vertex set of G_D is $\{1,2,\dots 10\}$, and two vertices $i > j$ are adjacent if $(i-j) \in D$ or $(10+j-i) \in D$.*

Notice that $G_{\{1,2,3,4,5\}}$ is the complete graph K_{10}, and that the complement of G_D is $G_{\{1,2,3,4,5\}\setminus D}$. Some of the graphs are isomorphic.

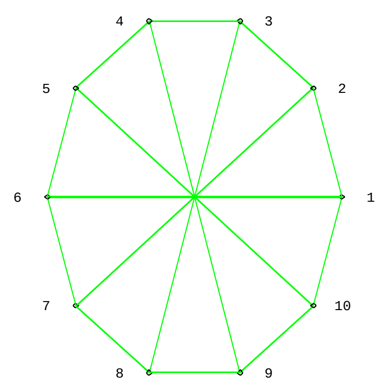


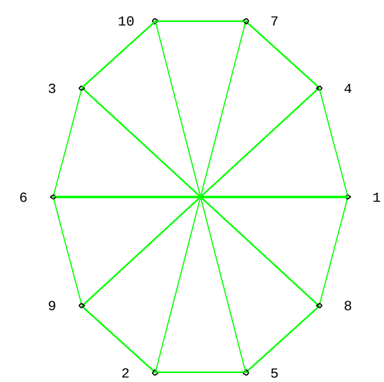
$G_{\{1\}} \simeq G_{\{3\}}$ and $G_{\{2,3,4,5\}} \simeq G_{\{1,2,4,5\}}$.



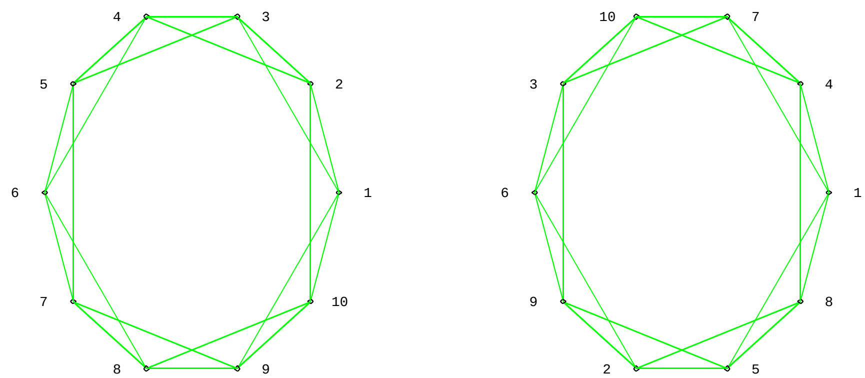


$G_{\{2\}} \simeq G_{\{4\}}$ and $G_{\{1,2,3,5\}} \simeq G_{\{1,3,4,5\}}$.

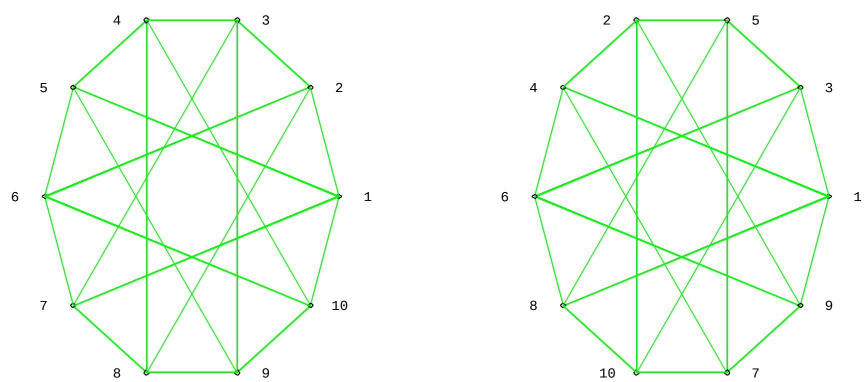




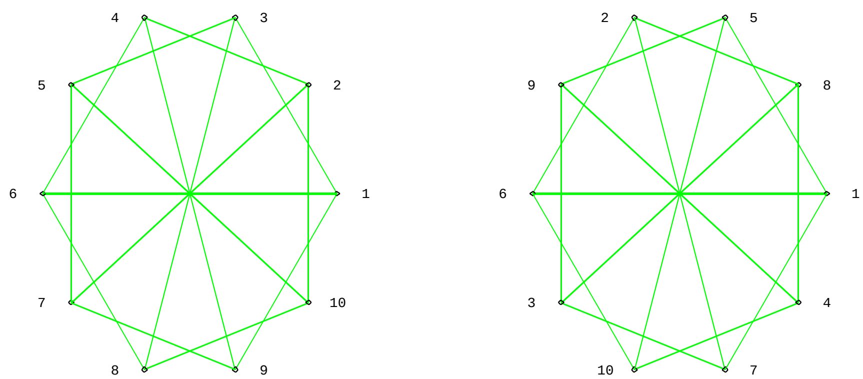
$G_{\{1,5\}} \simeq G_{\{3,5\}}$ and $G_{\{1,2,4\}} \simeq G_{\{2,3,4\}}$.



$G_{\{1,2\}} \simeq G_{\{3,4\}}$ and $G_{\{1,2,5\}} \simeq G_{\{3,4,5\}}$.



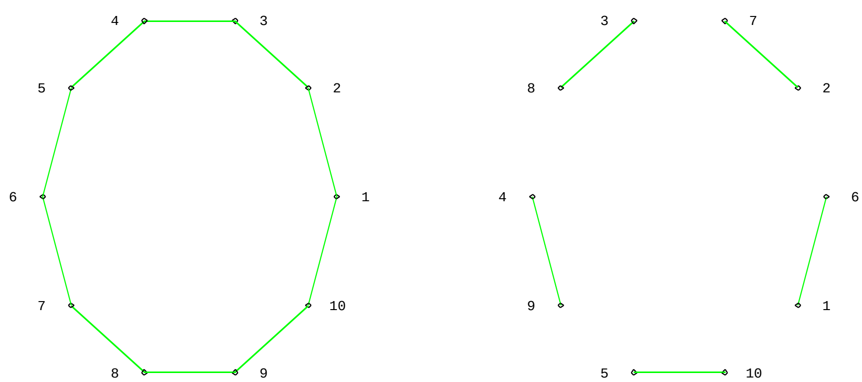
$G_{\{1,4\}} \simeq G_{\{2,3\}}$ and $G_{\{1,4,5\}} \simeq G_{\{2,3,5\}}$.



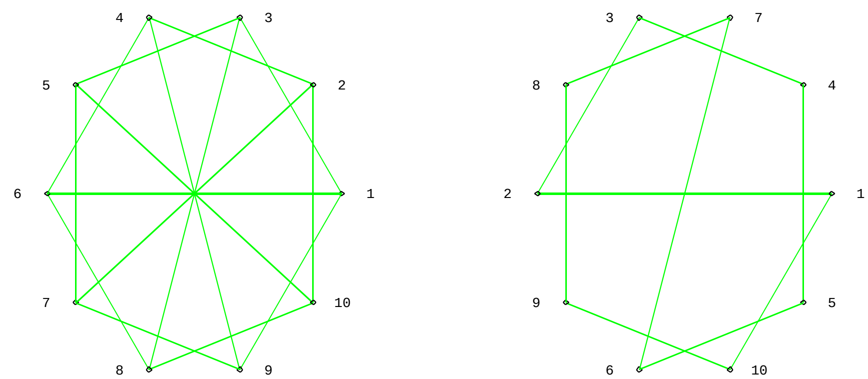
$G_{\{2,5\}} \simeq G_{\{4,5\}}$ and $G_{\{1,2,3\}} \simeq G_{\{1,4,5\}}$.

2.1.2 Inclusion poset

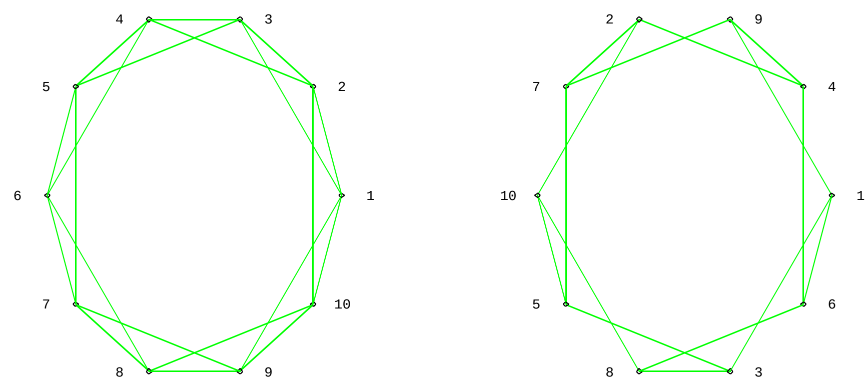
The graph properties we study are monotone, so it is important to know which graphs are subgraphs of others. If $D \subset D' \subseteq \{1, 2, 3, 4, 5\}$, then obviously $G_D < G_{D'}$, but otherwise?



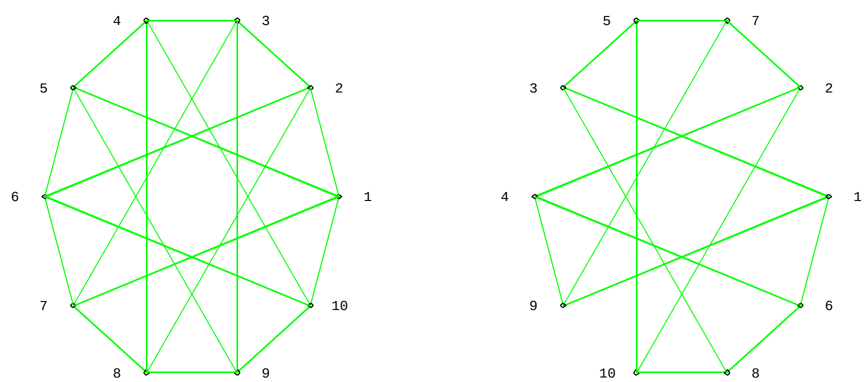
$G_{\{1\}} \simeq G_{\{3\}} > G_{\{5\}}$ and $G_{\{1,2,3,4\}} > G_{\{2,3,4,5\}} \simeq G_{\{1,2,4,5\}}$.



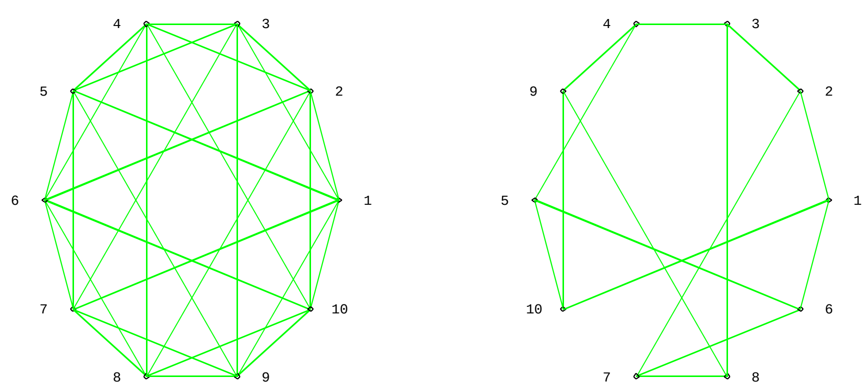
$G_{\{2,5\}} \simeq G_{\{4,5\}} > G_{\{1\}} \simeq G_{\{3\}}$ and
$G_{\{2,3,4,5\}} \simeq G_{\{1,2,4,5\}} > G_{\{1,2,3\}} \simeq G_{\{1,3,4\}}$.



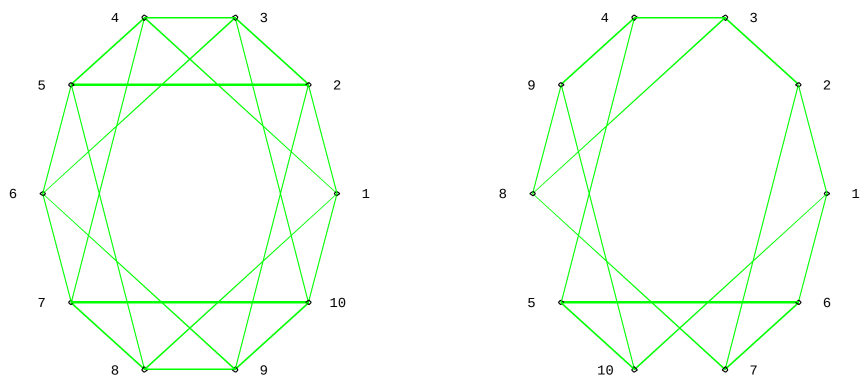
$G_{\{1,2\}} \simeq G_{\{3,4\}} > G_{\{2,5\}} \simeq G_{\{4,5\}}$ and
$G_{\{1,2,3\}} \simeq G_{\{1,3,4\}} > G_{\{1,2,5\}} \simeq G_{\{3,4,5\}}$.



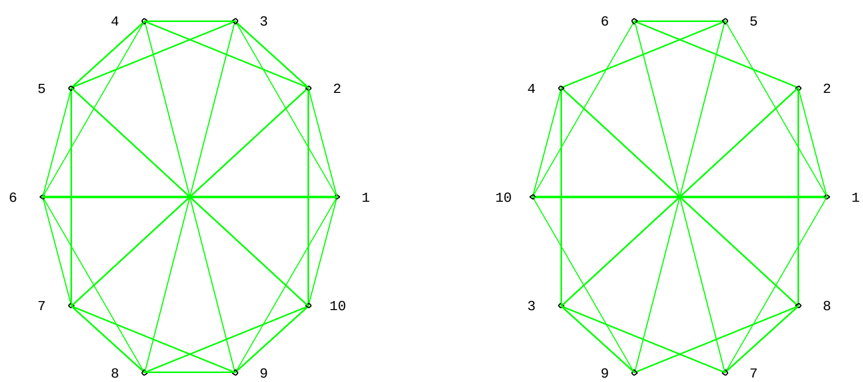
$G_{\{1,4\}} \simeq G_{\{2,3\}} > G_{\{2,5\}} \simeq G_{\{4,5\}}$ and
$G_{\{1,2,3\}} \simeq G_{\{1,3,4\}} > G_{\{1,4,5\}} \simeq G_{\{2,3,5\}}$.



$G_{\{1,2,4\}} \simeq G_{\{2,3,4\}} > G_{\{1,5\}} \simeq G_{\{3,5\}}$

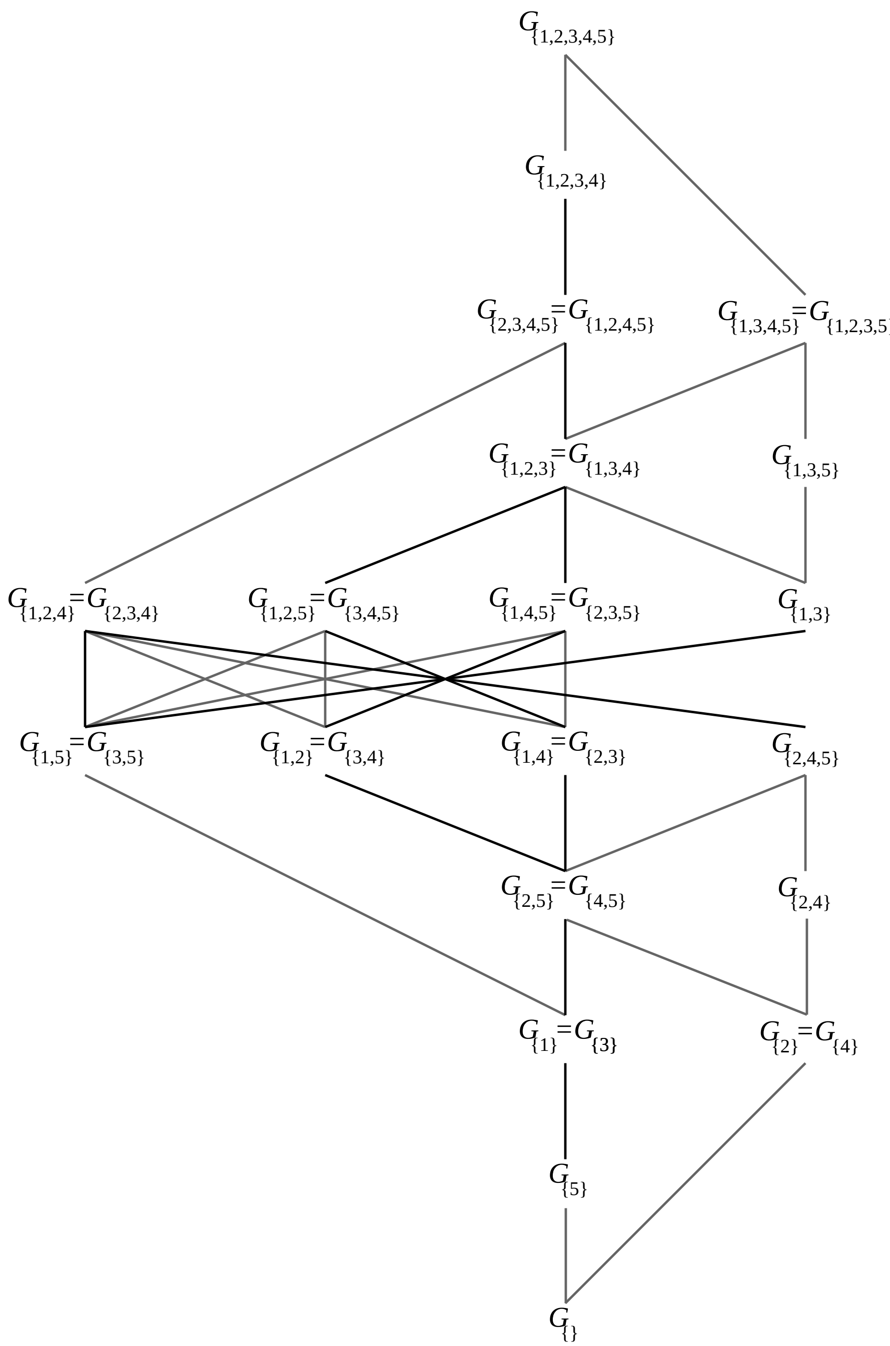


$G_{\{1,3\}} > G_{\{1,5\}} \simeq G_{\{3,5\}}$ and $G_{\{1,2,4\}} \simeq G_{\{2,3,4\}} > G_{\{2,4,5\}}$.



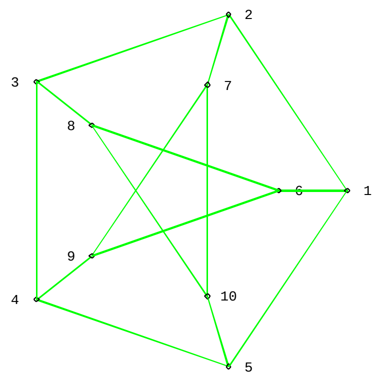
$G_{\{1,2,5\}} \simeq G_{\{3,4,5\}} > G_{\{1,4\}} \simeq G_{\{2,3\}}$ and
$G_{\{1,4,5\}} \simeq G_{\{2,3,5\}} > G_{\{1,2\}} \simeq G_{\{3,4\}}$.

The inclusion relation give a poset which is drawn in the next figure. Grey lines are trivial inclusions, and black lines are inclusions listed above.

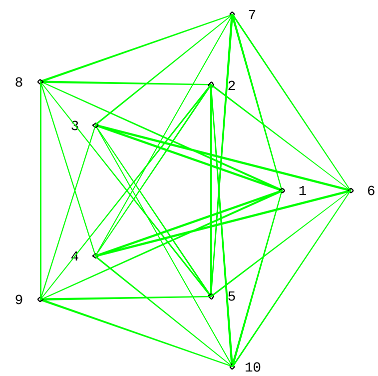


2.2 The Petersen graph

The non Cayley transitive graphs are the Petersen graph and its complement.

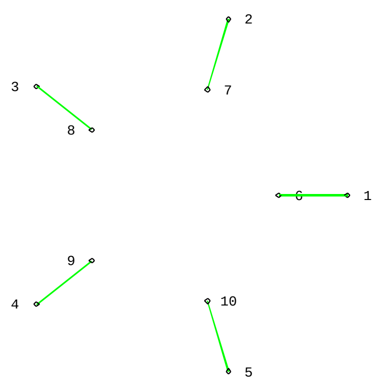


The Petersen graph P.

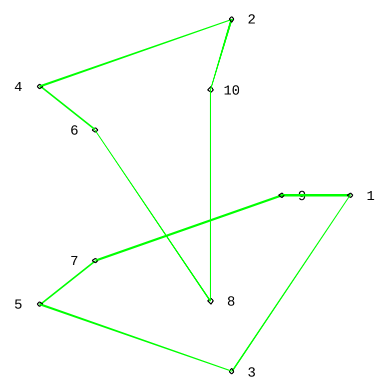


The complement of the Petersen graph, $\bar{P}$.

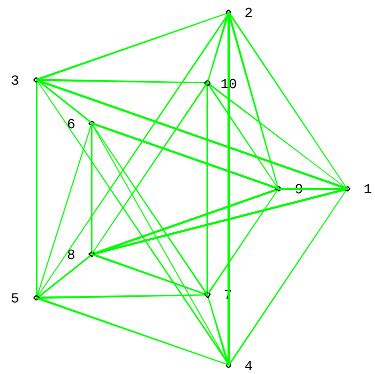
From the figure above it is clear that $P < \bar{P}$. The other relations are:



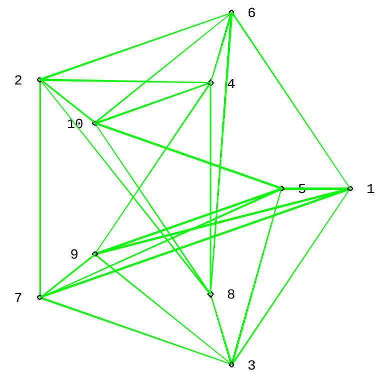
$G_{\{5\}} < P$
$\bar{P} < G_{\{1,2,3,4\}}$



$G_{\{2\}} \simeq G_{\{4\}} < P$
$\bar{P} < G_{\{1,2,3,5\}} \simeq G_{\{1,3,4,5\}}$

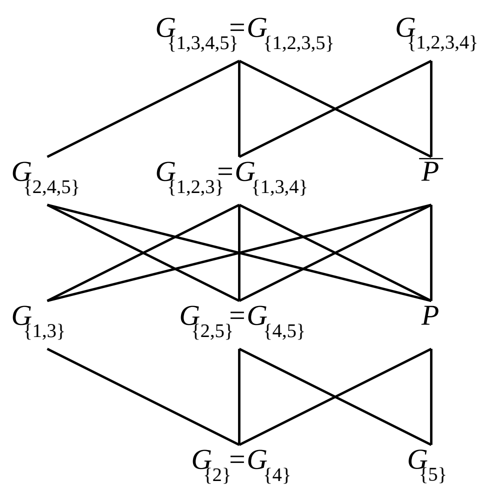


$P < G_{\{1,2,3\}} \simeq G_{\{1,3,4\}}$
$G_{\{2,5\}} \simeq G_{\{4,5\}} < \bar{P}$



$P < G_{\{2,4,5\}}$
$G_{\{1,3\}} < \bar{P}$

Inserting P and $\bar{P}$ in the inclusion poset of the Cayley graphs would be a mess. But selecting the elements with a cover relation to P and $\bar{P}$ gives the poset:



3 Using the topological method

If G is a graph on 10 vertices, and Δ a simplicial complex, then define the indicator i_G as: $i_G = 1$ if $G \in \Delta$ and $i_G = 0$ if $G \notin \Delta$. To simplify notation for the Cayley graphs, we write i_{124} instead of $i_{G_{\{1,2,4\}}}$ for example. Note that by monotonicity $i_G \leq i_{G'}$ if $G \supseteq G'$. The two posets in the last section carries over to the indicators, and that is how we will use them.

Let $\mathfrak{C}$ be the set of nonevasive nontrivial graph complexes on 10 vertices. Note that $i_{12345} = 0$ for all nontrivial complexes.

Lemma 8 *For any simplicial complex in $\mathfrak{C}$, $i_5 = 1$ and $i_{1234} = 0$.*

PROOF: Let

- $\Gamma = \langle (1\,3\,5\,7\,9)(2\,4\,6\,8\,10), (1\,7\,9\,3)(2\,4\,8\,6), (2\,7)(5\,10) \rangle$.
- Define a homomorphism ϕ from Γ onto $\mathbb{Z}_4$ by $\phi((1\,3\,5\,7\,9)(2\,4\,6\,8\,10)) = 0$, $\phi((1\,7\,9\,3)(2\,4\,8\,6)) = 1$ and $\phi((2\,7)(5\,10)) = 0$.
- $\Gamma' = \mathrm{Ker}(\phi) = \langle (1\,3\,5\,7\,9)(2\,4\,6\,8\,10), (2\,7)(5\,10) \rangle$.
- Define a homomorphism ϕ' from Γ' onto $\mathbb{Z}_5$ by $\phi'((1\,3\,5\,7\,9)(2\,4\,6\,8\,10)) = 3$, and $\phi'((2\,7)(5\,10) = 0$.
- $\Gamma'' = \mathrm{Ker}(\phi') = \langle (2\,7)(5\,10), (4\,9)(5\,10), (3\,8)(5\,10), (1\,6)(2\,7)(3\,8)(5\,10) \rangle$.

Then

- $\Gamma'' \triangleleft \Gamma' \triangleleft \Gamma$,
- $|\Gamma''| = 2^4$,
- $\Gamma'/\Gamma'' = \mathbb{Z}_5$ is cyclic,
- $|\Gamma'/\Gamma''| = |\mathbb{Z}_4| = 2^2 \Rightarrow q = 2$.

By theorem 5, $\chi(\Delta^\Gamma) \equiv 1 \bmod q$. The vertices of Δ^Γ are $\{G_{\{1,2,3,4\}}, G_{\{5\}}\}$, and $G_{\{5\}} < G_{\{1,2,3,4\}}$, so $i_5 = 1$ and $i_{1234} = 0$ is the only way to achive $\chi(\Delta^\Gamma) = i_5 + i_{1234} \equiv 1 \bmod 2$. □

Lemma 9 *For any simplicial complex in $\mathfrak{C}$,*

$$\begin{aligned} &2i_1 + 2i_2 - 2i_{12} - i_{13} - 2i_{14} - 2i_{15} - i_{24} - 2i_{25} + 2i_{123} \\ &+2i_{124} + 2i_{125} + i_{135} + 2i_{145} + i_{245} - 2i_{1235} - 2i_{1245} = 0 \end{aligned}$$

PROOF: Let $\Gamma = \langle (1\,2\,3\,4\,5\,6\,7\,8\,9\,10) \rangle$ and $\Gamma' = \mathrm{Id}$. By theorem 4, $\chi(\Delta^\Gamma) = 1$. The vertex set of Δ^Γ is $\{G_{\{1\}}, G_{\{2\}}, G_{\{3\}}, G_{\{4\}}, G_{\{5\}}\}$, so

$$\begin{aligned} \chi(\Delta^\Gamma) &= i_1 + i_2 + i_3 + i_4 + i_5 \\ &\quad -i_{12} - i_{13} - i_{14} - i_{15} - i_{23} - i_{24} - i_{25} - i_{34} - i_{35} - i_{45} \\ &\quad +i_{123} + i_{124} + i_{125} + i_{134} + i_{135} + i_{145} + i_{234} + i_{235} + i_{245} + i_{345} \\ &\quad -i_{1234} - i_{1235} - i_{1245} - i_{1345} - i_{2345} \\ &= 2i_1 + 2i_2 + 1 \\ &\quad -2i_{12} - i_{13} - 2i_{14} - 2i_{15} - i_{24} - 2i_{25} \\ &\quad +2i_{123} + 2i_{124} + 2i_{125} + i_{135} + 2i_{145} + i_{245} \\ &\quad -0 - 2i_{1235} - 2i_{1245} \\ &= 1 \end{aligned}$$

□

Lemma 10 *For any simplicial complex in $\mathfrak{C}$,*

$$2i_2 + i_{13} - i_{24} - 2i_{25} - 2i_{123} - i_{135} + i_{245} + 2i_{1235} = 0$$

Proof: Let

- $\Gamma = \langle (1\,3\,5\,7\,9)(2\,4\,6\,8\,10), (1\,2\,9\,8)(3\,6\,7\,4)(5\,10) \rangle$.
- Define a homomorphism ϕ from Γ onto $\mathbb{Z}_4$ by $\phi((1\,3\,5\,7\,9)(2\,4\,6\,8\,10)) = 0$, and $\phi((1\,2\,9\,8)(3\,6\,7\,4)(5\,10)) = 1$.
- $\Gamma' = \mathrm{Ker}(\phi) = \langle (1\,3\,5\,7\,9)(2\,4\,6\,8\,10) \rangle$.

Then

- $\Gamma' \lhd \Gamma$,
- $\Gamma/\Gamma' = \mathbb{Z}_4$ is cyclic.
- $|\Gamma'| = 5$ is a prime power.

The vertex set of Δ^Γ is $\{G_{\{1,3\}}, G_{\{2\}}, G_{\{4\}}, G_{\{5\}}\}$, so by theorem 4,

$$\begin{aligned}
\chi(\Delta^\Gamma) &= i_{13} + i_2 + i_4 + i_5 - i_{123} - i_{134} - i_{135} - i_{24} - i_{25} - i_{45} + i_{1234} + i_{1235} + i_{1345} + i_{245} \\
&= i_{13} + 2i_2 + 1 - 2i_{123} - i_{135} - i_{24} - 2i_{25} + 0 + 2i_{1235} + i_{245} \\
&= 2i_2 + i_{13} - i_{24} - 2i_{25} - 2i_{123} - i_{135} + i_{245} + 2i_{1235} + 1 \\
&= 1
\end{aligned}$$

$\square$

Lemma 11 *For any simplicial complex in $\mathfrak{C}$,*

$$2i_2 - i_{24} + i_{135} - 2i_{1235} = 1$$

Proof: Let

- $\Gamma = \langle (2\,4\,6\,8\,10), (1\,6)(2\,7)(3\,8)(4\,9)(5\,10) \rangle$.
- Let ϕ be a homomorphism from Γ onto $\mathbb{Z}_{10}$ by $\phi((2\,4\,6\,8\,10)) = 4$ and $\phi((1\,6)(2\,7)(3\,8)(4\,9)(5\,10)) = 5$.
- $\Gamma' = \mathrm{Ker}(\phi) = \langle (1\,3\,5\,7\,9)(2\,10\,8\,6\,4) \rangle$.

Then

- $\Gamma' \lhd \Gamma$,
- $\Gamma/\Gamma' = \mathbb{Z}_{10}$ is cyclic.
- $|\Gamma'| = 5$ is a prime power.

The vertex set of Δ^Γ is $\{G_{\{1,3,5\}}, G_{\{2\}}, G_{\{4\}}\}$, so by theorem 4,

$$\begin{aligned}\chi(\Delta^\Gamma) &= i_{135} + i_2 + i_4 - i_{24} - i_{1235} - i_{1345} \\ &= 2i_2 - i_{24} + i_{135} - 2i_{1235} \\ &= 1\end{aligned}$$

$\square$

Lemma 12 *For any simplicial complex in $\mathfrak{C}$,*

$$i_{14} = i_{145}$$

Proof: Let

- $\Gamma = \langle(1\,3\,5\,7\,9)(2\,4\,6\,8\,10), (2\,7)(5\,10)\rangle$.
- Let ϕ be the homomorphism from Γ onto $\mathbb{Z}_5$ defined by $\phi((13579)(246810)) = 1$, and $\phi((2\,7)(5\,10)) = 0$.
- $\Gamma' = \mathrm{Ker}(\phi) = \langle(1\,6)(2\,7), (1\,6)(5\,10)(3\,8)(4\,9), (4\,9)(5\,10)\rangle$.

Then

- $\Gamma' \triangleleft \Gamma$,
- $\Gamma/\Gamma' = \mathbb{Z}_5$ is cyclic.
- $|\Gamma'| = 2^4$ is a prime power.

The vertex set of Δ^Γ is $\{G_{\{1,4\}}, G_{\{2,3\}}, G_{\{5\}}\}$, so by theorem 4,

$$\begin{aligned}\chi(\Delta^\Gamma) &= i_{14} + i_{23} + i_5 - i_{145} - i_{235} - i_{1234} \\ &= 2i_{14} + 1 - 2i_{145} - 0 \\ &= 1\end{aligned}$$

$\square$

This lemma can be directly deduced from lemma 11, but it is not clear which one could be generalized most.

Lemma 13 *For any simplicial complex in $\mathfrak{C}$,*

$$i_{24} + i_{135} = 1$$

Proof: Let

- $\Gamma = \langle(1\,7\,9\,3)(2\,4\,8\,6), (2\,4\,6\,8\,10), (1\,4\,3\,2\,9\,6\,7\,8)(5\,10)\rangle$.
- Let ϕ be the homomorphism from Γ onto $\mathbb{Z}_8$ by $\phi((1\,7\,9\,3)(2\,4\,8\,6)) = 2$, $\phi((2\,4\,6\,8\,10)) = 0$, and $\phi((1\,4\,3\,2\,9\,6\,7\,8)(5\,10)) = 3$.
- $\Gamma' = \mathrm{Ker}(\phi) = \langle(2\,4\,6\,8\,10), (1\,5\,9\,3\,7)(2\,10\,8\,6\,4)\rangle$.

Then

- $\Gamma' \triangleleft \Gamma$,
- $\Gamma/\Gamma' = \mathbb{Z}_8$ is cyclic.
- $|\Gamma'| = 5^2$ is a prime power.

The vertex set of Δ^Γ is $\{G_{\{1,3,5\}}, G_{\{2,4\}}\}$, so by theorem 4,

$$\chi(\Delta^\Gamma) = i_{135} + i_{24} = 1.$$

$\square$

4 Gathering the facts

The indicators $i_3, i_4, i_{23}, i_{34}, i_{35}, i_{45}, i_{134}, i_{234}, i_{235}, i_{345}, i_{1245}, i_{1234}$, and i_{1235} are equal to another indicator by graph isomorphism. By lemma 8, $i_\emptyset = i_5 = 1$ and $i_{1234} = i_{12345} = 0$. We have the equalities

Lemma	Equality
9	$2i_1 + 2i_2 - 2i_{12} - i_{13} - 2i_{14} - 2i_{15} - i_{24} - 2i_{25} + 2i_{123} +$ $2i_{124} + 2i_{125} + i_{135} + 2i_{145} + i_{245} - 2i_{1235} - 2i_{1245} = 0$
10	$2i_2 + i_{13} - i_{24} - 2i_{25} - 2i_{123} - i_{135} + i_{245} + 2i_{1235} = 0$
11	$2i_2 - i_{24} + i_{135} - 2i_{1235} = 1$
12	$i_{14} = i_{145}$
13	$i_{24} + i_{135} = 1$

Theorem 14 *For any simplicial complex in $\mathfrak{C}$, the indicators of the transitive graphs are as one of the following six columns.*

Indicator\Name	**A**	**A***	**B**	**B***	**C**	**C***
i_1	1	1	1	1	1	1
i_2	0	1	1	1	1	1
i_5	1	1	1	1	1	1
i_{12}	0	1	1	1	0	1
i_{13}	1	1	0	0	0	0
i_{14}	0	1	0	1	0	1
i_{15}	1	1	0	1	1	1
i_{24}	0	0	1	1	1	1
i_{25}	0	1	1	1	1	1
i_{123}	0	1	0	0	0	0
i_{124}	0	0	0	1	0	0
i_{125}	0	1	0	0	0	1
i_{135}	1	1	0	0	0	0
i_{145}	0	1	0	1	0	1
i_{245}	0	0	1	1	1	1
i_{1234}	0	0	0	0	0	0
i_{1235}	0	1	0	0	0	0
i_{1245}	0	0	0	0	0	0
i_P	0	1	1	1	1	1
$i_{\bar{P}}$	0	1	0	0	0	0

PROOF: The proof is in three cases.
Case 1: $i_2 = 0$.

The indicators of all graph including $G_{\{2\}}$ are zero. The only undetermined are i_1, i_{15}, i_{13}, and i_{135}. By lemma 13, $i_{24} + i_{135} = 1 \Rightarrow i_{135} = 1$. Since $i_1 \geq i_{15} \geq i_{13} \geq i_{135}$, they are all equal to 1. Since $i_2 = 0$, $i_P = i_{\bar{P}} = 0$. This is column **A**.

Case 2: $i_{1235} = 1$.

This is the dual assumption of $i_2 = 0$, which gives column $\mathbf{A}^*$.

Case 3: $i_2 = 1$ and $i_{1235} = 0$.

Lemma 11 becomes $-i_{24} + i_{135} = -1$, hence $i_{24} = 1$ and $i_{135} = 0$. Lemma 10 becomes $i_{13} - 2i_{25} - 2i_{123} + i_{245} = -1$. Both i_{25} and i_{123} cannot be 1, and $i_{123} \leq i_{25}$, so $i_{123} = 0$. And now $i_{25} = 1$ since $i_{13} - 2i_{25} + i_{245} = -1$. Thus $i_{13} + i_{245} = 1$. Note that $i_1 = 1$ and $i_{1245} = 0$ since $i_{25} = 1$ and $i_{123} = 0$.

Case 3.1: $i_{13} = 1$ and $i_{245} = 0$.

From subgraph inclusion we get $i_{15} = 1$ and $i_{124} = 0$. Remove i_{14} and i_{145} from lemma 9 by using lemma 12, and insert values to get $-2i_{12} + 2i_{124} = 2$. But this can never be true since $i_{12} \geq i_{124}$.

Case 3.2: $i_{13} = 0$ and $i_{245} = 1$.

Once again removing i_{14} and i_{145} from lemma 9 by using lemma 12, and insert values, gives $i_{12} + i_{15} - i_{124} - i_{125} = 1$. Since both i_{12} and i_{15} are greater or equal with both i_{124} and i_{125}, we have four different options, which are the columns $\mathbf{B}, \mathbf{B}^*, \mathbf{C}, \mathbf{C}^*$. The value of $i_{14} = i_{145}$ is uniquely determined in each column since $i_{12}, i_{15} \geq i_{145} = i_{14} \geq i_{124}, i_{125}$. Finally about the Petersen graph: $i_P \geq i_{245} = 1$ and $i_{\bar{P}} \leq i_{13} = 0$. □

5 Computational aspects

The proofs in this paper are computer independent but to find the right lemmas for section 3 several algorithms were implemented in MAPLE. First a library of subgroups of S_{10} satisfying the conditions of Theorem 4 and 5 was constructed. This library, of what is called non Oliver groups, was not complete. Each group gives a linear equation of indicators, and only those with indicators of transitive graphs were used. This system of equations was heavily linearly dependent, so almost all equations with their groups were thrown away. Those left became the lemmas of section 3.

One interesting continuation of this work is the removal of conditions on the indicators. That would give a huge equation system, but maybe sufficient conditions on the graph properties to do the remaining search for a counterexample by brute force.

References

[1] A. Björner, Topological methods, in: R. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, North-Holland, Amsterdam, 1995, pp. 1819–1872.

[2] J. Kahn, M. Saks, D. Sturtevant *A topological approach to evasiveness* Combinatorica **4** (1984), no. 4, 297–306.

[3] D.N. Kozlov *Collapsing along monotone poset maps*, preprint, `math.CO/0503416`.

[4] H. Kurzweil *A combinatorial technique for simplicial complexes and some applications to finite groups*, Discrete Math. **82** (1990) 263-278.

[5] F.H. Lutz *Examples of $\mathbb{Z}$-acyclic and contractible vertex-homogeneous simplicial complexes* Discrete Comput. Geom. **27** (2002), no. 1, 137–154.

[6] F.H. Lutz *Some results on the evasiveness conjecture* J. Combin. Theory Ser. B **81** (2001), no. 1, 110–124.

[7] B.D. McKay *Transitive graphs with fewer than twenty vertices.* Math. Comp. **33** (1979), no. 147, 1101–1121.

[8] R. Oliver *Fixed-Point Sets of Group Actions* Comment. Math. Helvetici **50** (1974), 155–177.

[9] G. Royle *Transitive graphs.* `http://www.csse.uwa.edu.au/~gordon/remote/trans/index.html`

[10] P.A. Smith *Fixed point theorems for periodic transformations*, Amer. J. of Math. **63** (1941), 1–8.

[11] V. Welker *Constructions preserving evasiveness and collapsibility*, Discrete Math. **207** (1999), no. 1–3, 243–255.